WEAKLY SUPERCRITICAL CONVECTION

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1. Introduction

The experimental and theoretical investigation of convection began relatively recently. At the start of this century in the first experiments in a horizontal layer with weak supercriticality Bénard observed the formation of a spatially periodic hexagonal structure. The linear theory of this phenomenon, convective instability, was already understood by Rayleigh. As for the study of a nonlinear regime, a sufficiently consistent theory of this phenomenon was constructed relatively recently [1, 2].

According to this theory, hexagonal cells form owing to a weak dependence of the viscosity η on the temperature T. In particular, from this theory there followed the conclusion, confirmed experimentally, that the direction of convective circulation is determined by the sign of $\partial\eta/\partial T$, while the excitation of cells takes place strictly up to an amplitude proportional to $\partial\eta/\partial T$ (for more detail on this see [3] and the literature cited there). When such a dependence of the viscosity on the temperature is absent, one-dimensional periodic structures form: rollers. And the excitation of rollers, as shown by subsequent experiments [4] using Doppler velocity meters, takes place mildly in complete accordance with the Landau law [5]. It should be noted that from the first experiments of Bénard the formation of cells has been observed in those cases when the upper surface is free. But when the upper surface is rigid one observes rollers at a weak supercriticality, as a rule, while hexagonal cells, developing due to the weak dependence of the viscosity on the temperature and existing, according to [2], in a small range of supercriticalities, are observed rather rarely for this reason.

In the present report it is shown that at a weak supercriticality the effects connected with a free surface are decisive in the formation of hexagonal cells in a number of cases. The formation of such cells represents an analog of a phase transition. This fully pertains to any transition from a laminar to a turbulent state. For example, a phase transition of the second kind corresponds to a mode of mild excitation while a phase transition of the first kind corresponds to a mode of hard excitation. We emphasize that the transition to weakly supercritical convection in a horizontal layer is two dimensional. The latter is connected with the fact that unstable states are characterized by a wave vector $\mathbf{k}$ lying in the horizontal plane and a discrete number n, which in the simplest case coincides with the number of half-waves vertically. Therefore, at a weak supercriticality perturbations with the minimum number n build up; because of isotropy in the horizontal plane their increment γ_k is positive in a narrow layer near $|\mathbf{k}| = k_0$ ($\gamma|_{k_0} = 0$). This instability is aperiodic, and therefore three-particle processes will be important in the nonlinear stage. Perturbations whose wave vectors form an angle of $\pi/3$ are connected with each other in these processes. It is fundamental that three-particle processes do not stabilize an instability. This can be understood from the following. Let us consider three excited modes with equal and real amplitudes A, the wave vectors of which just form an angle of $\pi/3$. Then the evolution of A is determined from the equation

$$\frac{\partial A}{\partial t} = \gamma A + \frac{1}{2} U A^2,$$

simple integration of which leads to

$$A = \left[\left(\frac{1}{A_0} + \frac{U}{2\gamma}\right) e^{-\gamma t} - \frac{U}{2\gamma}\right]^{-1}.$$

From this it is seen that when $U > 0$ there is a time when the amplitude goes to infinity. This is a so-called explosive instability. When $U < 0$ a singularity also develops, only for perturbations differing from those discussed by a phase shift of π. Thus, three-particle processes lead only to a correlation of three interacting modes; these processes do not remove the degeneracy with respect to angle and do not stop the instability. Stabilization of an instability can only be provided by a four-particle interaction. It is clear that in describing such an interaction near the threshold one must be confined only to the region near $|\mathbf{k}| = k_0$, since far from it the perturbations die out: $\gamma_k < 0$. On this basis the equation for the amplitudes A_k of the excited perturbations has the form

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$$\frac{\partial A_k}{\partial t} = \gamma_k A_k + \frac{U}{2}\int A_{k_1}A_{k_2}\delta_{k-k_1-k_2}dk_1dk_2 - \frac{1}{3!}\int T_{-kk_1k_2k_3}A_{k_1}A_{k_2}A_{k_3}\delta_{k-k_1-k_2-k_3}dk_1dk_2dk_3, \quad (1.1)$$

where U and $T_{kk_1k_2k_3}$ are matrix elements of three-particle and four-particle interactions taken at surfaces $|\mathbf{k}_i| = k_0$, and $A_k = A^*_{-k}$. We note that Eq. (1.1) presumes that the nonlinearity is small. This actually means that a matrix element U at a resonance surface must have a smallness not connected with the supercriticality. For example, this criterion is invalid for supercritical flows forming as a result of the development of a thermocapillary instability, due to the dependence of the coefficient of surface tension α on the temperature. In particular, all the quantitative results of [6], in which this effect is analyzed, are therefore erroneous.

It should be noted that this problem, formulated in terms of A_k, is similar to the problem which we considered on the development of hexagonal relief at the surface of a liquid dielectric when a vertical field is turned on [7].

In the present report it is shown that in weakly supercritical convection the matrix element U differs from zero owing to two weak effects: the effect of finite deformation of the free surface and the thermocapillary effect with weak nonisothermicity of the free surface. It is just this circumstance which provides for the existence of stable hexagonal cells with weak supercriticality. With large supercriticalities the cells become unstable, while one-dimensional structures (rollers) acquire stability.

2. Basic Equations

To describe convection we use the dimensionless Boussinesq equations [3] for the velocity and the perturbation of the temperature T, reckoned from the equilibrium value $T_0 = -Az + B$ $(A > 0)$:

$$\mathrm{Pr}^{-1}d\mathbf{v}/dt = -\nabla p + \Delta\mathbf{v} + \mathrm{Ra}T\mathbf{e}_z; \quad (2.1)$$

$$\partial T/\partial t + (\mathbf{v}\nabla)T = \Delta T + v_z, \ \mathrm{div}\,\mathbf{v} = 0, \quad (2.2)$$

where $\mathrm{Ra} = \beta g A h^4/\nu\chi$ is the Rayleigh number; $\mathrm{Pr} = \nu/\chi$ is the Prandtl number; $\mathbf{e}_z$ is the unit vector directed along the z axis; β is the coefficient of thermal expansion; ν and χ are the coefficients of viscosity and thermal diffusivity; h is the size along the z axis. In these equations time is measured in units of h^2/χ, velocity in units of χ/h, and temperature in units of Ah.

Henceforth we will take $\mathrm{Pr} \gg 1$. For water, for example, Pr = 5, while for oils the Prandtl number reaches 10^2 and sometimes 10^3. Therefore, the inertial term in Eq. (2.1) can be neglected.

Now let us formulate the boundary conditions. We will distinguish two types of boundary conditions at the lower boundary. The first, the so-called Rayleigh boundary conditions, are

$$v_z = 0, \ \partial\mathbf{v}_\perp/\partial z = 0, \ T = 0 \ \text{at} \ z = 0, \quad (2.3)$$

while the second are those with a solid boundary:

$$\mathbf{v} = 0, \ T = 0 \ \text{at} \ z = 0. \quad (2.4)$$

At the free deformable surface $z = 1 + \zeta(\mathbf{r}_\perp, t)$ the first boundary condition represents the kinematic connection

$$\partial\zeta/\partial t = v_z - (\mathbf{v}_\perp\nabla_\perp)\zeta; \quad (2.5)$$

the second represents the equality of forces

$$\left[-\zeta + W\Delta_\perp\zeta - \frac{\mu\mathrm{Ra}}{2}\zeta^2\right]n_i = \mu\left[-p\delta_{ik} + \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right]n_k; \quad (2.6)$$

and the third represents the condition of isothermicity

$$T|_{z=1+\zeta} = \zeta, \quad (2.7)$$

where $\mathbf{n}$ is the normal to the surface; $W = \alpha/\rho g h^2$; $\mu = \nu\chi/gh^3$. From these boundary conditions it is seen that in the limit $\mu \to 0$ the conditions (2.5)-(2.7) change into the Rayleigh boundary conditions.

For layers which are not very narrow the parameter μ is actually small, which one can ascertain by representing μ in the form

$$\mu = (\gamma_g/\omega_g)^2\mathrm{Pr}^{-1},$$

where $\gamma_g = \nu/h^2$ and $\omega_g = (g/h)^{1/2}$ are the damping and the frequency of a gravitational surface wave with $k \sim h^{-1}$. The ratio ω_g/γ_g appearing here represents the quality of the waves, which is high, as a rule. For this reason the parameter μ is small.

Another possible effect which exists at a free surface is connected with the dependence of the coefficient of surface tension α on the temperature [8]. Naturally, this effect is possible only when the free surface is nonisothermal. The nonisothermicity of the surface can be modeled by the interpolation condition [3]

$$-\varkappa \partial T/\partial n = a(T - T_2),$$

where $\varkappa$ is the coefficient of thermal conductivity; T_2 is the temperature of the upper mass as $z \to \infty$.

We will assume that the nonisothermicity is weak, which corresponds to the condition $b = ah/\varkappa \gg 1$. Therefore, it is clear that in the thermocapillary effect one should not allow for the deformation of the free surface, which is characterized by another small parameter μ.

Then taking $\alpha = \alpha_0 - \sigma T$ and changing to perturbations, we arrive at the following boundary conditions:

$$v_z = 0, \quad \frac{\partial v_\perp}{\partial z} = -B\nabla_\perp T, \quad T = \frac{1}{b}\frac{\partial T}{\partial z} \quad \text{at} \quad z = 1,$$

where $B = A\sigma h^2/\rho\nu\chi$. We emphasize that as $b \to \infty$ these boundary conditions once again change into the Rayleigh boundary conditions.

Let us turn to the calculation of γ, U, and $T_{kk_1k_2k_3}$. First we will briefly discuss the linear theory of convective instability. We note that the operator

$$L\begin{pmatrix} \mathbf{v} \\ T \end{pmatrix} = \begin{pmatrix} -\nabla p + \Delta \mathbf{v} + \mathrm{Ra}\, T\mathbf{e}_z \\ \Delta T + v_z \end{pmatrix} = L\Psi$$

with $\mu = 0$ and $b = \infty$ is self-adjoint if the scalar product is defined in the form

$$(\Psi_1, \Psi_2) = \int (\mathrm{Ra}\, T_1^* T_2 + \mathbf{v}_1^* \mathbf{v}_2)\, dr.$$

From this it follows, in particular, that the instability is aperiodic. It is obvious that this stability also remains aperiodic when $\mu \ll 1$ and $b \gg 1$.

Assuming Rayleigh boundary conditions at the lower surface and $\mu = 0$ and $b = \infty$, we write the expression for the increment (cf. [3]) as

$$\gamma_{kn} = \frac{k^2(\mathrm{Ra} - \mathrm{Ra}_{kn})}{(k^2 + \pi^2 n^2)^2} \tag{2.8}$$

for the temperature eigenfunctions $\Theta_{kn} = \sin n\pi z$, and for the velocity

$$u_{zkn} = \frac{\mathrm{Ra}\, k^2}{(k^2 + \pi^2 n^2)^2} \sin n\pi z, \quad \mathbf{u}_{\perp kn} = \frac{i\mathbf{k}}{k^2}\frac{\partial u_{zkn}}{\partial z},$$

where $\mathrm{Ra}_{kn} = (k^2 + n^2\pi^2)^3/k^2$.

From Eq. (2.8) it follows that the instability threshold is reached at $n = 1$, $k = k_0 = \pi/\sqrt{2}$, and $\mathrm{Ra}_c = 27\pi^4/4$. Therefore, near the instability threshold $\gamma \sim \mathrm{Ra} - \mathrm{Ra}_c$. We note that the increment behaves exactly the same way when the lower surface is solid. According to [3], in this case the instability threshold is reached at $n = 1$, $k_0 = 2.682$, and $\mathrm{Ra}_c = 1100, 657$. Then the neutral eigenfunction has the form

$$\Theta_{k1} = \sin \varkappa_1 (1 - z) - 2 \sin \varkappa_1 \mathrm{Re}\left[e^{-i\pi/3}\frac{\sin \varkappa_2 (1 - z)}{\sin \varkappa_2}\right],$$

where

$$\varkappa_1 = 3.569; \quad \varkappa_2 = 1.895 + i4.555; \quad (\varkappa_{1,2}^2 + k_0^2)^3 = k_0^2 \mathrm{Ra}_c.$$

Henceforth the eigenfunctions of the linear problem with $\mathrm{Ra} = \mathrm{Ra}_c$, $k = k_0$, $\mu = 0$, and $b = \infty$ will be designated as Θ_0 and u_0.

Now let us proceed to the calculation of the matrix elements. First we consider the case of $\mu = 0$ and $b = \infty$. In this limit, as was noted, the operator L is self-adjoint, while the boundary conditions are linear. Therefore, the sole nonlinear term is the term $(\mathbf{v}\nabla)T$ in Eq. (2.1). Then expanding Ψ by eigenfunctions of the linear problem,

$$\Psi = \sum_n \int \psi_{kn}(r_\perp) A_{kn}(t)\, dk \quad (A_{kn}^* = A_{-kn}),$$

we arrive at the equation

$$\frac{\partial A_{kn}}{\partial t} = \gamma_{kn} A_{kn} + \frac{1}{2} \sum_{n_1 n_2} \int U_k^n \Big|_{k_1 k_2}^{n_1 n_2} A^*_{k_1 n_1} A^*_{k_2 n_2} \delta_{k+k_1+k_2} dk_1 dk_2.$$

From the self-adjoint nature of the operator L and $v_z = 0$ at $z = 0.1$ we get the important result that the matrix element $U_{kk_1k_2}$ is equal to zero at the surface $|\mathbf{k}_i| = k_0$. In fact, by virtue of the identity

$$0 = \int (\mathbf{v}\nabla) \frac{T^2}{2} dr = -2 \sum_{nn_1 n_2} \int \Big(U_k^n \Big|_{k_1 k_2}^{n_1 n_2} A^*_{kn} A^*_{k_1 n_1} A^*_{k_2 n_2} + \text{c.c.} \Big) \delta_{k+k_1+k_2} dk_1 dk_2 dk$$

and the arbitrariness of A_{kn}, the following relations develop for the matrix elements:

$$\left(U_k^n \Big|_{k_1 k_2}^{n_1 n_2} + U_{k_1}^{n_1} \Big|_{k k_2}^{n n_2} + U_{k_2}^{n_2} \Big|_{k_1 k}^{n_1 n} \right) \delta_{k+k_1+k_2} = 0.$$

Taking $n = n_1 = n_2 = 1$ and $|\mathbf{k}_i| = k_0$ in this equation, we arrive at $U = 0$. Thus, three-particle terms are absent in the expansion (1.1) with $\mu = 0$ and $b = \infty$. We emphasize that this conclusion is also valid when the upper surface is solid. The latter means that the expansion in (1.1) starts with the four-particle term. This in turn leads to a mild mode of excitation, in full accordance with experiment [4] and with the Landau theory [5, 9].* Therefore, in the Boussinesq approximation the matrix element U can be different from zero only because of the free surface. Here we can distinguish two factors: first, the nonlinearity of the boundary conditions, and second, the non-self-adjoint nature of the operator L. Since $\mu \ll 1$ and $b \gg 1$, the two effects (the effect of a deformable surface and the thermocapillary effect) can be considered separately.

3. Calculation of the Matrix Elements

First we calculate the contribution to the matrix element U due to the finite surface deformation. First of all we make several simplifications. We recall that the matrix element U is defined near the surface $|\mathbf{k}_i| = k_0$, i.e., in the three-particle interaction the wave vectors of the perturbations form a regular triangle with good accuracy. Therefore, for example, $(v_\perp \nabla_\perp) T \simeq \frac{1}{2} \frac{\partial v_z}{\partial z} T$ when $z = 1$. It is also obvious that the matrix element U due to a finite deformation is proportional to μ. Therefore, it is necessary to neglect the term $\partial\zeta/\partial t$ in (2.5). Then expanding all the functions with $z = 1 + \zeta$ in series with respect to ζ, we arrive at the following boundary conditions:

$$v_z = -\frac{1}{2} \frac{\partial v_z}{\partial z} T; \tag{3.1}$$

$$(-1 + W\Delta_\perp)\left(T + T\frac{\partial T}{\partial z}\right) = \mu\left(-p + 2\frac{\partial v_z}{\partial z}\right); \tag{3.2}$$

$$\nabla_\perp v_z + \frac{\partial v_\perp}{\partial z} = -2\frac{\partial v_z}{\partial z}\nabla_\perp T - T\frac{\partial^2 v_\perp}{\partial z^2}, \tag{3.3}$$

which, in an approximation linear with respect to amplitude, have the form

$$v_z = 0, \quad \frac{\partial v_\perp}{\partial z} = 0, \quad (-1 + W\Delta_\perp) T = \mu\left(-p + 2\frac{\partial v_z}{\partial z}\right). \tag{3.4}$$

With such linear boundary conditions the operator L is no longer self-adjoint, i.e., the boundary conditions at the free surface for the eigenfunction $\overline{\Psi}$ of the conjugate problem no longer coincide with the analogous ones for Ψ. They are determined from the equation

$$(\overline{\Psi}, L\Psi) - (L\,\overline{\Psi}, \Psi) = 0.$$

One can verify that the difference between these integrals is reduced to an integral over the surface $z = 1$, from which one can obtain

$$(-1 + W\Delta_\perp)\bar{u}_z = \mu \mathrm{Ra}\, \partial\overline{\Theta}/\partial z, \ \overline{\Theta} = 0, \ \partial\bar{u}_\perp/\partial z + \nabla_\perp \bar{u}_z = 0. \tag{3.5}$$

The boundary conditions for the conjugate problem at the lower surface are the same as for the direct problem.

Now let us proceed to the derivation of the equations for the amplitudes A_k. We note that in these equations the matrix element T develops in the second order with respect to $U_k^n\Big|_{k_1 k_2}^{n_1 n_2}$, which is not connected with

*It should be emphasized that for convection the sign of the matrix element T is positive, as will be shown below.

TABLE 1

Boundary conditions at z=0	$U_d\left(1+Wk_0^2\right)$	μRa	$\frac{U_t b}{B}$	T_0	$T_{\pi/6}$	$T_{\pi/4}$	$T_{\pi/3}$	$T_{\pi/2}$
$v_z=0,\ \frac{dv_\perp}{dz}=0$	3,202		−0,205	5,552	5,153	4,753	4,351	3,950
$v_z=0,\ v_\perp=0$	3,207		−0,225	10,225	9,364	8,506	7,569	6,810

the small parameters μ and b^{-1} whereas the matrix element U is proportional to them. Therefore, at supercriticalities $(\mathrm{Ra}-\mathrm{Ra_c})^{1/2}\sim\mu$, b^{-1} all the terms in (1.1) prove to be of the same order. Thus, the solution of Eqs. (2.1) and (2.2) near the threshold must be sought in the form of an asymptotic series by powers of the supercriticality,

$$\Psi = \Psi_1 + \delta\Psi,$$

where Ψ is the exact solution of the linear problem; $\delta\Psi$ is a higher-order perturbation.

Then expanding v and T by eigenfunctions of the linear problem, from (2.1) and (2.2) we arrive at the equation

$$\mathrm{Ra}\,\partial A_k/\partial t\langle\overline{\Theta}_k|\Theta_k\rangle = \overline{(\psi}_k,\, L\Psi) - \mathrm{Ra}\langle\overline{\Theta}_k|(\mathbf{v}_1\nabla T_1)_k\rangle - \mathrm{Ra}\langle\overline{\Theta}_k|(\mathbf{v}_1\nabla\delta T)_k\rangle - \mathrm{Ra}\langle\overline{\Theta}_k|(\delta\mathbf{v}\nabla T_1)_k\rangle,$$

where $\langle|\rangle$ denotes integration with respect to z.

In this equation the last two terms after the iteration basically correspond to a four-particle interaction not containing the small parameters μ and b^{-1}. In particular, the last term is basically equal to zero at the surface $|\mathbf{k}_i| = k_0$. Therefore, when finding δT and $\delta\mathbf{v}$ from the equations

$$0 = -\nabla\delta p + \Delta\delta\mathbf{v} + \mathrm{Ra}\,\delta T\mathbf{e}_z,\ \mathbf{v}_1\nabla T_1 = \Delta\delta T + \delta v_z \tag{3.6}$$

it is sufficient to use the Rayleigh boundary conditions at z = 1:

$$\delta v_z = 0,\quad \frac{\partial^2}{\partial z^2}(\delta v_z) = 0,\quad \delta T = 0.$$

If we apply the rot rot operator to the first equation of (3.6) and change to the Fourier components, then we obtain the equations

$$(\mathbf{v}_1\nabla T_1)_k = \frac{1}{2\pi}\int g(\varphi)\,A_{k_1}A_{k_2}\delta_{k-k_1-k_2}dk_1dk_2,$$

$$\delta T_k = \frac{1}{2\pi}\int \tau(\varphi)\,A_{k_1}A_{k_2}\delta_{k-k_1-k_2}dk_1dk_2,$$

where the function

$$g(\varphi) = \frac{1-\cos\varphi}{2}\frac{\partial}{\partial z}(u_{0zk}\Theta_{0k}) + \frac{1+\cos\varphi}{2}\left(u_{0zk}\frac{\partial\Theta_{0k}}{\partial z} - \frac{\partial u_{0zk}}{\partial z}\Theta_{0k}\right)$$

is connected with $\tau(\varphi)$ by the equations

$$0 = -\Delta^2 w_z + k^2\mathrm{Ra}\,\tau,\ g = \Delta\tau + w_z \tag{3.7}$$

with the boundary conditions (2.3), (2.4), and (3.4) with $\mu = 0$, while φ is the angle between the vectors k_1 and k_2 ($k^2 = 2k_0^2(1+\cos\varphi)$) and $\Delta = \partial^2/\partial z^2 - k^2$.

We note that in the case of a Rayleigh lower boundary a particular solution of the system (3.7) satisfies the boundary conditions. And in the case of a solid lower surface the corresponding solution of the homogeneous system must be added to the particular solution.

Then substituting δT into the integral $-\mathrm{Ra}\langle\overline{\Theta}_k|(\mathbf{v}_1\nabla\delta T)_k\rangle$, we obtain

$$-\frac{\mathrm{Ra}}{(2\pi)^2}\int I(\varphi_{12})\,A_{k_1}A_{k_2}A_{k_3}\delta_{k-k_1-k_2-k_3}dk_1dk_2dk_3,$$

where $I(\varphi) = -\int g(\varphi)\,\tau(\varphi)\,dz$; φ_{12} is the angle between the vectors k_1 and k_2.

A matrix element of the four-frequency interaction is obtained from this by symmetrization over all the wave vectors. For Rayleigh boundary conditions at the surface $|\mathbf{k}_i| = k_0$ the matrix element is

$$T_\varphi = T_{k_1k_2-k_1-k_2} = \frac{9\pi^2}{32}\left[1 + \frac{4(5+\cos\varphi)^2(1-\cos\varphi)^2}{4(5+\cos\varphi)^3 - 27(1+\cos\varphi)} + \frac{4(5-\cos\varphi)^2(1+\cos\varphi)^2}{4(5-\cos\varphi)^3 - 27(1-\cos\varphi)}\right].$$

The values of T_φ for a solid lower boundary are given in Table 1.

Now let us turn to the calculation of the matrix element U. As indicated above, it is different from zero because of the nonlinear boundary conditions and the non-self-adjoint nature of the operator L. The contribution of the first effect is determined from the integral

$$(\bar{\psi}_k, L\Psi) = (L\bar{\psi}_k, \Psi) + (\bar{\psi}_k, L\Psi) - (L\bar{\psi}_k, \Psi),$$

where the first term leads to a term linear with respect to amplitude,

$$\mathrm{Ra}\,\gamma_k\langle\bar{\Theta}_k|\Theta_k\rangle A_k,$$

while the difference between the last two terms comes down to an integral over the surface z = 1 and is different from zero by virtue of the nonlinear boundary conditions (3.1)-(3.3). Using the explicit expression for them, we can reduce this difference to the form

$$\int dr_\perp\left[\mathrm{Ra}\frac{\partial\Theta_0^*}{\partial z}T\frac{\partial T}{\partial z} - \frac{1}{2k_0^2}\frac{\partial u_{0z}^*}{\partial z}\frac{\partial^3 v_z}{\partial z^3}T - \frac{1}{2}p_0^*\frac{\partial v_z}{\partial z}T\right].$$

In this expression we expand T and v_z in series with respect to the eigenfunctions of the linear problem. Finally, the contribution to the matrix element U due to the nonlinear boundary conditions has the form

$$U_N = \frac{\mu\left(3u'_{0zk} - \frac{1}{k_0^2}u'''_{0zk}\right)}{\pi\,\mathrm{Ra}\,\langle\Theta_{0k}|\Theta_{0k}\rangle\left(1+Wk_0^2\right)}\left\{\frac{1}{2}u'_{0zk}\left(\frac{2}{k_0^2}u'''_{0zk} - u'_{0zk}\right) - \mathrm{Ra}_c\left(\Theta'_{0k}\right)^2\right\}\Big|_{z=1}.$$

Now let us consider the contribution to U due to the integral $-\mathrm{Ra}\langle\bar{\Theta}_k|(\mathbf{v}_1\nabla T_1)_k\rangle$. When $\mu = 0$ this integral is equal to zero by virtue of the self-adjoint nature of the operator L (see Sec. 2). Therefore, we expand the eigenfunctions of the direct and conjugate prpblems in series with respect to μ. Then being confined to terms of first order with respect to μ ($\delta\bar{\psi}$, $\delta\psi \sim \mu$), we obtain

$$-\mathrm{Ra}\langle(\delta\bar{\Theta}_k - \delta\Theta_k)|(\mathbf{u}_0\nabla\Theta_0)_k\rangle\int A_{k_1}A_{k_2}\delta_{k-k_1-k_2}dk_1dk_2.$$

The expression $(\mathbf{u}_0\nabla\Theta_0)_k$ appearing here is precisely the function $g(\varphi)$ which we introduced with $\varphi = 2\pi/3$, so that the matrix element is

$$-\mathrm{Ra}\langle(\delta\bar{\Theta}_k - \delta\Theta_k)|(\mathbf{u}_0\nabla\Theta_0)_k\rangle = (\delta\bar{\psi}_k - \delta\psi_k, L\Phi),$$

where $\Phi = (\tau, w_z)$ is the solution of the system (3.7) with $\varphi = 2\pi/3$. Integration of this matrix element by parts leads to a sum of surface terms when z = 1. Then using the boundary conditions (3.4) and (3.5), we arrive at the expression

$$U_s = \frac{\mu}{\pi\langle\Theta_{0k}|\Theta_{0k}\rangle\left(1+Wk_0^2\right)}\left\{\Theta'_{0k}\left(3w'_z - \frac{w'''_z}{k_0^2}\right) - \tau'\left(3u'_{0zk} - \frac{u'''_{0zk}}{k_0^2}\right)\right\}\Big|_{z=1}.$$

Thus, the matrix element U due to the finite deformation is

$$U_d = U_N + U_s;$$

in particular, with Rayleigh boundary conditions at the lower surface we have

$$U_d = \frac{1124\,\mu\,\mathrm{Ra}_0}{351\left(1+Wk_0^2\right)}.$$

The value of U_d for a solid lower boundary is given in Table 1.

The matrix element due to the thermocapillary effect is calculated similarly:

$$U_t = \frac{B}{\pi b\,\mathrm{Ra}\,\langle\Theta_{0k}|\Theta_{0k}\rangle}\left(\Theta'_{0k}w'_z - \tau' u'_{0zk}\right)\big|_{z=1} \tag{3.8}$$

(see Table 1). The total matrix element U due to the free surface consists of the sum $U_d + U_t$.

Let us give the expression for the matrix element U_η connected with the temperature dependence of the coefficient of viscosity:

$$U_\eta = \frac{2\xi}{\mathrm{Ra}\,\langle\Theta_{0k}|\Theta_{0k}\rangle}\int\left(\frac{\partial u_{0i}}{\partial x_l}\right)_k^*\left[\Theta_0\left(\frac{\partial u_{0i}}{\partial x_l} + \frac{\partial u_{0l}}{\partial x_i}\right)\right]_k dz, \tag{3.9}$$

where $\xi = -\frac{d\ln\eta}{dT}Ah.$

4. Steady-State Solutions and Their Stability

Now let us proceed to a study of steady-state solutions, being confined to the consideration only of periodic solutions with vectors q of the reciprocal lattice whose lengths are equal to the critical value k_0.

By virtue of the two-dimensionality, Eq. (1.1) has only three steady-state periodic solutions, in the form of hexagonal cells

$$A_k = A_3 \sum_{i=1}^{3} (\delta_{k-q_i} + \delta_{k+q_i}), \quad \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0,$$

square cells

$$A_k = A_2 (\delta_{k-q_1} + \delta_{k+q_1} + \delta_{k-q_2} + \delta_{k+q_2}), \quad (\mathbf{q}_1\mathbf{q}_2) = 0,$$

and rollers

$$A_k = A_1(\delta_{k-q} + \delta_{k+q}),$$

the respective amplitudes of which are

$$A_3 = \frac{U}{4T_{\pi/3} + T_0} + \operatorname{sgn} U \left[\frac{2\gamma_{k_0}}{4T_{\pi/3} + T_0} + \left(\frac{U}{4T_{\pi/3} + T_0}\right)^2\right]^{1/2},$$

$$A_2 = \left(\frac{2\gamma_{k_0}}{2T_{\pi/2} + T_0}\right)^{1/2}, \quad A_1 = \left(\frac{2\gamma_{k_0}}{T_0}\right)^{1/2}.$$

The first solution is characterized by a hard mode of excitation with a jump at Ra = $\mathrm{Ra_c}$ having a size

$$A_3 = 2U/(4T_{\pi/3} + T_0),$$

while the other two are characterized by excitation with a mild mode:

$$A_{1,2} \sim \sqrt{\mathrm{Ra} - \mathrm{Ra_c}}.$$

The stability of these modes is determined from the linearized equation (1.1) for the amplitudes a_k:

$$\frac{\partial a_k}{\partial t} = \gamma_k a_k + U \int A_{k_1} a_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 - \frac{1}{2} \int T_{-kk_1k_2k_3} A_{k_1} A_{k_2} a_{k_3} \delta_{k-k_1-k_2-k_3} dk_1 dk_2 dk_3. \tag{4.1}$$

As in [7], we are confined to the consideration of perturbations $a_k(t) = a_k e^{\Gamma t}$ with wave numbers on the order of k_0 ($k - k_0 \sim \sqrt{\mathrm{Ra} - \mathrm{Ra_c}}$), which are the most dangerous from the point of view of stability. For such perturbations

$$\gamma_k = c\varepsilon - f(k - k_0)^2,$$

where

$$c = \frac{\langle u_{0zk} | \Theta_{0k} \rangle}{\langle \Theta_{0k} | \Theta_{0k} \rangle}; \quad \varepsilon = \mathrm{Ra} - \mathrm{Ra_c}; \quad f = \frac{c}{2} \frac{\partial^2 \mathrm{Ra}_k}{\partial k^2}.$$

First let us study the stability of rollers in detail. Here and later we distinguish three regions for perturbations a_k whose wave vectors lie in the layer of $k - k_0 \sim \sqrt{\varepsilon}$. In the first region the angle φ between the perturbation vector k and the vector of the reciprocal lattice is not close to any of the values of 0, $\pi/3$, $2\pi/3$, or π (external stability, nonresonance perturbations), in the second region the vector k is close to one of the vectors of the reciprocal lattice q and −q (internal stability), and in the third region the angle φ is close to $\pi/3$ or $2\pi/3$, when the perturbations are resonantly connected with each other by a three-particle interaction.

In the first case the eigenmodes are plane waves, for which the dispersion equation has the form

$$\Gamma = \gamma_k - T_\varphi A_1^2,$$

where φ is the angle between the vectors k and q. Rollers are stable relative to such perturbations.

In the second region the eigenfunctions of Eq. (4.1) represent combinations of two perturbations:

$$a_k = a_1 \delta_{k-q-\varkappa} + a_{-1} \delta_{k+q-\varkappa}.$$

The system of equations for the amplitudes a_1 and a_{-1} is decomposed into two equations: for even ($c_+ = (a_1 + a_{-1})/2$) and odd ($c_- = (a_1 - a_{-1})/2i$) perturbations. Their eigenvalues are (cf. [10])

$$\Gamma_+ = -2c\varepsilon - f\varkappa^2 \cos^2\varphi < 0, \ \Gamma_- = -f\varkappa^2 \cos^2\varphi < 0,$$

where φ is the angle between the vectors $\varkappa$ and **q**.

In the third region the eigenfunctions also represent combinations of two perturbations,

$$a_k = a_1 \delta_{k-k_1-\varkappa} + a_2 \delta_{k-k_2-\varkappa}$$

($k_1 = k_2 + \mathbf{q}$), with the eigenvalues

$$\Gamma=\left(\frac{T_0}{2}-T_{\pi/3}\right)A_1^2-\frac{f\varkappa^2}{2}\left(1-\frac{1}{2}\cos 2\varphi\right)\pm\left[(UA_1)^2+\frac{3}{16}f^2\varkappa^4\sin^2 2\varphi\right]^{1/2}.$$

The condition

$$\Gamma_{\max}=\left(\frac{T_0}{2}-T_{\pi/3}\right)A_1^2+|U|A_1<0$$

determines the region of stability of rollers:

$$c(\mathrm{Ra}-\mathrm{Ra}_c)>\frac{2T_0}{(2T_{\pi/3}-T_0)^2}U^2=c\varepsilon_1.$$

The stabilities of square and hexagonal cells are investigated in a similar way.

Square cells always prove to be unstable. It is fundamental that they are unstable relative to perturbations with wave vectors from the second region; for these perturbations

$$\Gamma_{\max}=(2T_{\pi/2}-T_0)A_2^2>0.$$

As for hexagonal cells, they are stable relative to nonresonance perturbations

$$\Gamma=c\varepsilon-f(k-k_0)^2-(T_\varphi+T_{\varphi+\pi/3}+T_{\varphi-\pi/3})A_3^2<0$$

and to odd resonance perturbations

$$\Gamma=-f\varkappa^2/2\pm f\varkappa^2/4<0,\ \Gamma=-3UA_3-f\varkappa^2/2<0.$$

For even resonance perturbations the eigenvalues have the form

$$\Gamma=2c\varepsilon-2(T_0+T_{\pi/3})A_3^2-\frac{f\varkappa^2}{2}\pm\frac{f\varkappa^2}{4},$$

$$\Gamma=-c\varepsilon-\frac{1}{2}(T_0+4T_{\pi/3})A_3^2-\frac{f\varkappa^2}{2}.$$

The condition $\Gamma<0$ determines the region of stability of hexagonal cells

$$-\frac{1}{2}\frac{U^2}{T_0+4T_{\pi/3}}<c\varepsilon<4\frac{T_0+T_{\pi/3}}{(2T_{\pi/3}-T_0)^2}U^2=c\varepsilon_2.$$

From this it follows that in the range of supercriticalities $\varepsilon_1<\varepsilon<\varepsilon_2$ both hexagonal cells and rollers are stable. The transition from one state to another is hard.

Thus, for weak supercriticality the first bifuraction is the transition to hexagonal cells with a hard mode of excitation. This transition is due to a three-particle interaction, the matrix element of which differs from zero for a free upper boundary owing to the thermocapillary effect and the deformation of the surface. Naturally, with allowance for the temperature dependence of the viscosity all three of these effects make an additive contribution to U. The relative contributions of each are determined by the parameters U_t/U_η and U_d/U_η. For the first of them, in accordance with (3.8) and (3.9),

$$U_t/U_\eta\sim\left(\frac{\omega_\alpha}{\gamma_\alpha}\right)^2\frac{\mathrm{Pr}}{b}\frac{d\ln\alpha}{d\ln\eta},$$

where ω_α and γ_α are the frequency and the damping of a capillary wave with $k\sim h^{-1}$. From this it is seen that this ratio can be larger than unity owing to the factor $(\omega_\alpha/\gamma_\alpha)^2\mathrm{Pr}$. For water with h = 1 cm, for example, $U_t/U_\eta\approx 10^4/b$, i.e., these two effects are comparable at a nonisothermicity $b\approx 10^4$. It should be emphasized that the thermocapillary effect, characterized by the parameter B, influences the convective instability. In very narrow layers $h\ll h_c=(\sigma/\rho g\beta)^{1/2}$ this instability is reorganized and changes into a thermocapillary instability [9]. Therefore, our analysis is valid when $h\gg h_c$.

As for deformation effects, they are important when $d\ln\rho/d\ln\eta\approx 1$. For almost all liquids, however, this parameter is small, $\sim 10^{-1}$, 10^{-2}. It becomes on the order of unity in the vicinity of the inversion point T_c, where $\partial\eta/\partial T=0$. For sulfur, for example, T_c = 153°C.

LITERATURE CITED

1. E. Palm, "On the tendency towards hexagonal cells in steady convection," J. Fluid Mech., 8, No. 2 (1960).
2. F. H. Busse, "The stability of finite amplitude cellular convection and its relation to an extremum principle," J. Fluid Mech., 30, No. 4 (1967).
3. G. Z. Gershuni and E. M. Zhukhovitskii, Convective Stability of an Incompressible Fluid [in Russian], Nauka, Moscow (1972).

4. P. Berge and M. Dubois, "Convective velocity field in the Rayleigh–Bénard instability: experimental results," Phys. Rev. Lett., 32, 1041 (1974).
5. L. D. Landau, Collected Works [in Russian], Vol. 1, Nauka, Moscow (1969).
6. J. W. Scanlon and L. A. Segel, "Finite amplitude cellular convection induced by surface tension," J. Fluid Mech., 30, No. 1 (1967).
7. E. A. Kuznetsov and M. D. Spektor, "On the existence of hexagonal relief at the surface of a liquid dielectric in an external electric field," Zh. Eksp. Teor. Fiz., 71, No. 1 (1976).
8. J. K. A. Pearson, "On convective cells induced by surface tension," J. Fluid Mech., 4, No. 5 (1958).
9. L. P. Gor'kov, "Steady convection in a plane layer of liquid near a critical mode of heat transfer," Zh. Eksp. Teor. Fiz., 33, No. 2 (1957).
10. A. C. Newell and J. A. Whitehead, "Finite bandwidth, finite amplitude convection," J. Fluid Mech., 38, No. 2 (1969).

EXPERIMENTAL INVESTIGATION OF MIXED AIR CONVECTION NEAR A HORIZONTAL CYLINDER

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Heat exchange with mixed convection near a horizontal cylinder plays an important role in a number of technological processes. In addition, a cylinder is a convenient model for a fundamental investigation of the process. Several reports on this problem have now been published. The boundary-layer equations, written in the Boussinesq approximation, have been solved numerically for the region of a cylinder where the use of boundary-layer theory is possible [1-3]. The numerical investigation was carried out most fully in [1], where, along with results on heat exchange and friction, data were obtained on the influence of gravitational forces on the separation of the boundary layer. In all the reports the velocity distribution at the outer limit of the boundary layer is taken either from experimental data for purely forced motion or as for streamline flow of an ideal fluid, since there are no data for mixed convection. The average heat transfer at a constant wall temperature or at a constant heat flux is mainly considered in the experimental reports [4-7]. Only in [7] is the local heat exchange of a horizontal cylinder with a constant heat flux investigated for transverse flow over it. Data are absent on the hydrodynamic environment over the entire perimeter of a cylinder under conditions of mixed convection.

In the present report an experimental investigation is made of the flow of a vertical air stream over a horizontal isothermal cylinder when the directions of forced motion and the gravitational forces do and do not coincide. The influence of natural convection on the position of the separation point of the boundary layer is investigated. The velocity and temperature distributions are measured. The local and average heat fluxes are determined. The measurements are made at $Gr \approx 10^5$, Re = 40-4000, and Gr/Re^2 = 0.01-20.

The investigations were conducted in the working chamber of a vertical low-velocity wind tunnel which could operate in closed and open schemes. The stream velocity was varied in the range of 0-1 m/sec and the stream temperature was varied from 20 to 50°C. The degree of stream turbulence in the working chamber did not exceed 0.3%. A cylinder made of copper pipe 60 mm in diameter and 200 mm long was used as the working body. The degree of blockage of the stream by the cylinder was 0.12. As is known, such a level of turbulence and stream blockage does not affect the heat exchange for laminar flow over a cylinder. The cylinder was cooled or heated, depending on the required direction of natural convection.

The separation point of the boundary layer was determined through visualization of the flow by the method of a laser light "knife." A thin streamer of tobacoo smoke, which moved along a streamline in the boundary layer, was supplied in the plane of the "knife" in the vicinities of the upper or lower critical points of the cylinder. The separation point was accurately determined visually and from photographs from the sharp change in the direction of motion of the smoke streamer. The accuracy of determination of the angular coordinate of the separation point was no worse than 2° The velocity was measured with a laser anemometer of type 55L from

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